Shell model

The shell model explains the behaviour of a nucleon in the nucleus. According to this model, the protons and neutrons are grouped in shells in the nucleus similar to extranuclear electrons in various shells outside the nucleus. In the extra nuclear shells only one type of particles (i.e electrons) is to be arranged in different shells and Pauli's exclusion principle is applied. In the case of nucleus there are two types of particles protons and neutrons and the shell arrangement is only empirical and based upon the study of the stability and interactions of the nuclides which are known. This idea was first put forward by Elsasser that culminated in the development of nuclear shell model.

Points in favour of shell model:—

1. Mayer in 1948 suggested that nuclei with a magic numbers of nucleons i.e 2, 8, 20, 50, 82, 126 are especially abundant in nature.
2. $_2He^4$ and $_8O^{16}$ are particularly stable which can be seen from the binding energy curve. Thus the numbers 2, 8 indicate stability.
3. Above $Z = 28$, the only nuclides of even Z which have isotopic abundances exceeding 60% are Sr^{88} ($N = 50$), Ba^{138} ($N = 82$) and Ce^{140} ($N = 82$).
4. No more than five isotones occur in nature for any N except $N = 50$, where there are six and $N = 82$ where there are seven. Neutron numbers of 82, 50, therefore indicate particular stability.
5. Sn($Z = 50$) has ten stable isotopes, more than any other element while Ca($Z = 20$) has six isotopes. This indicates that elements with $Z = 50$ and $Z = 20$ are more than usually stable.
6. Alpha decay energies are rather smooth functions of A for a given Z but show striking discontinuities at $N = 126$. This represents the magic character of the number 126 for neutrons.
7. Very similar relations exist among the energies of beta-ray emission. These energies are very large when the neutron or proton number of the

②

final nucleus assumes a magic number.

8. The particularly weak binding of first nucleon outside a closed shell is & usually low probabilities for the capture of neutrons by nuclides having N=50, 80 and 126.

9. It is found that some isotopes are spontaneous neutron emitters when excited above the nucleon binding energy by a preceding β-decay. These are O (N = 9), Kr (N = 51) and Xe (N = 83).

10. Nuclei with the magic proton numbers 50 (Sn) and 82 (Pb) have much smaller capture cross-sections than their neighbours.

11. The doubly magic nuclei (Z and N both magic-numbers) $_2He^4$, $_8O^{16}$, $_{20}Ca^{40}$ and $_{82}Pb^{208}$ are particularly tightly bound.

12. The binding energy of the next neutron or proton after a magic number is very small.

13. The symmetry of the fission of uranium could involve the sub-structure of nuclei, which is expressed in the existence of the magic numbers.

14. The Schmidt theory of magnetic moments for odd A nuclides shows that the ground states of these nuclides change from even parity to odd parity or vice versa at the numbers A = 4, 16, 40, when the nucleon numbers are 2, 8 and 20 respectively.

15. The electric quadrupole moments of nuclei show sharp minima at the closed shell numbers, indicating that such nuclei are nearly spherical.

Salient features of shell model

This model assumes that each nucleon stays in a well-defined quantum state. But like the atom, the nucleus has no fixed centre of charge. The nucleons in the nucleus should frequently collide with one another due to high nuclear density. The concept of well-defined nucleonic orbits is difficult to comprehend. But Weisskopf

3.

pointed out that a nucleon may not be scattered if there is no empty quantum state to receive it.

In the shell model, therefore, each nucleon in the nucleus is considered as a single particle that moves independently of others in the time averaged field of the remaining (A−1) nucleons and is confined to its own orbit unperturbed by others. In terms of Schrodinger's equation, each nucleon thus moves in the same potential $V(r)$ which may be taken as an average harmonic oscillator potential so that $V(r) = \frac{1}{2}Kr^2$. Schrodinger equation then becomes

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}Kr^2\right)\psi = E\psi$$

having the solution:

$$E_n = \left(N + \frac{1}{2}\right)\hbar\omega$$

where N = oscillator quantum number $= 0, 1, 2, 3, \cdots$

The wavefunction ψ has both angular and the radial part.

Each nucleon is supposed to have an orbital angular momentum $|\vec{\lambda}| = \sqrt{l(l+1)}\,\hbar$ where $l = 0, 1, 2, 3, \cdots$ the nuclear orbital quantum number. Another quantum number, very similar to the principal quantum number of electronic orbit characterises the radial part of nuclear wave function and is symbolised by $n = 1, 2, 3, \cdots$. Each nucleon has also spin angular momentum $|\vec{s}| = \sqrt{s(s+1)}\,\hbar$ where $s = \frac{1}{2}$ and behaves as an independent particle subject to Pauli's principle that no two identical nucleons can be in the same quantum state. Again as suggested by Mayor, Jensen and others, there is a strong interaction between orbital and intrinsic spin angular momenta of each nucleon. The quantum mechanical rules for angular momenta dictate that total angular momentum $j\hbar$ formed by the vector addition of orbital angular momentum $l\hbar$ and spin $s\hbar$ must be such that j is restricted to the following two values:

$j = l + \frac{1}{2}$ and $j = l - \frac{1}{2}$.

Thus a different energy is associated with each ~~other~~ of the two j-levels and each nucleonic energy level with a given l splits into two sub-levels, except for $l = 0$, when j has only one value $\frac{1}{2}$. The level $j = l + \frac{1}{2}$ corresponds to $\vec{s}$ and $\vec{l}$ parallel ($\uparrow\uparrow$) to each other and $j = l - \frac{1}{2}$ to $\vec{s}$ and $\vec{l}$ anti-parallel ($\uparrow\downarrow$) to each other. Empirically, it is found that the nuclear energy level with higher j always lies below that with smaller j. So $j = l + \frac{1}{2}$ sub-level has a lower energy than $j = l - \frac{1}{2}$ sub-level. The former giving a more tightly bound nucleonic state. The separation between two sub-levels with a given l is rather large and increases rapidly with l. When $l \geq 4$, the sub-levels lie in different shells. The quantum number j of nucleus is obtained by Σj. To designate the nucleonic states, spectroscopic-notation of atomic physics is followed. Each sub-level can have a maximum of $(2j+1)$ nucleons of the same kind for a given j. So it can have $(2j+1)$ protons and $(2j+1)$ neutrons. Nucleons are designated with n-values followed by spectroscopic notation of l-values (s, p, d, f, --- for $l = 0, 1, 2, 3, \cdots$); the j-values are shown as subscript and the superscript gives the number of nucleons required to complete the sub-shell shown in the table as:

n	l	j	Designation and no. of p or n to fill sub-levels		Progressive Total
1	0	1/2	$\rightarrow (1s_{1/2})^2$	2	2
1	1	3/2	$\rightarrow (1p_{3/2})^4$		
1	1	1/2	$\rightarrow (1p_{1/2})^2$	6	8
1	2	5/2	$\rightarrow (1d_{5/2})^6$		
2	0	1/2	$\rightarrow (2s_{1/2})^2$		
1	2	3/2	$\rightarrow (1d_{3/2})^4$	12	20
1	3	7/2	$\rightarrow (1f_{7/2})^8$		
2	1	3/2	$\rightarrow (2p_{3/2})^4$		
1	3	5/2	$\rightarrow (1f_{5/2})^6$		
2	1	1/2	$\rightarrow (2p_{1/2})^2$		
1	4	9/2	$\rightarrow (1g_{9/2})^{10}$	30	50
1	4	7/2	$\rightarrow (1g_{7/2})^8$		
2	2	5/2	$\rightarrow (2d_{5/2})^6$		
2	2	3/2	$\rightarrow (2d_{3/2})^4$		
3	0	1/2	$\rightarrow (3s_{1/2})^2$		
1	5	11/2	$\rightarrow (1h_{11/2})^{12}$	32	82
1	5	9/2	$\rightarrow (1h_{9/2})^{10}$		
2	3	7/2	$\rightarrow (2f_{7/2})^8$		
2	3	5/2	$\rightarrow (2f_{5/2})^6$		
3	1	3/2	$\rightarrow (3p_{3/2})^4$		
3	1	1/2	$\rightarrow (3p_{1/2})^2$		
1	6	13/2	$\rightarrow (1i_{13/2})^{14}$	44	126

For instance, for $l = 0$, $j = l + \frac{1}{2} = \frac{1}{2}$ and number of nucleons in the level $= 2j+1 = 2 \times \frac{1}{2} + 1 = 2$ and the state is designated along with n-value as $(1s_{1/2})^2$. Similarly for $l = 1$, $j = l \pm s = 1 \pm \frac{1}{2} = \frac{3}{2}$ and $\frac{1}{2}$. The number of nucleons in the two sub-levels are thus $2 \times \frac{3}{2} + 1 = 4$ and $2 \times \frac{1}{2} + 1 = 2$. The two sub-states are designated as $(1p_{3/2})^4$ and $(1p_{1/2})^2$ respectively. So the total number of nucleons in this ~~Sub~~ level $= 4 + 2 = 6$, giving the progressive total of 8 nucleons and so it goes on.

Success and limitations

The shell model of nuclei has both its success and limitations.

Some of the Successes are:

1. It very well explains the existence of magic-numbers and the stability and high binding energy on the basis of closed shells.
2. The shell model provides explanation for the ground state spins and magnetic moments of the nuclei. The neutrons and protons with opposite spins pair off so that the mechanical and magnetic moment cancel and the odd or left out proton or neutron contributes to the spin and magnetic moment of the nuclei as a whole.
3. Nuclear isomerism i.e existence of isobaric, isotopic nuclei in different energy states of odd-A nuclei between 39-49, 69-81, 111 to 125 has been explained by shell model by the large difference in nuclear spins of isomeric states as their A-values are close to magic numbers.

Some of the limitations of shell model are:

1. The model does not predict the correct value of spin quantum number in certain nuclei for e.g $_{11}Na^{23}$ where the predicted value is 5/2 while the correct value is $\frac{1}{2}$.
2. The following four stable nuclei, $_1H^2$, $_3Li^6$, $_5B^{10}$ and $_7N^{14}$ do not fit into this model.
3. The model can not explain the observed first

excited states in even-even nuclei at energies much lower than those expected from single-particle excitation. It also fails to explain the observed large quadrupole moment of odd-A-nuclei, in particular of those having A-values far away from the magic numbers.